Math 142 Lecture 17 Notes

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1 Classification of 0- and 1-Manifolds

Many thanks to Jiabao Yang, who provided me with his notes, since I missed this lecture.

1.1 Classification of 0-manifolds

Theorem 1.1 (Generalized Poincarè Conjecture)**.** *If X is a closed, connected n-manifold, then $X \simeq S^n \implies X \cong S^n$.*

We will prove the cases $n = 1, 2$ by classifying such 1- and 2-manifolds.

Theorem 1.2. *All connected 0-manifolds are homeomorphic to $\{0\}$.*

Proof. If X is a 0-manifold, then for each $x \in X$, there exists and open neighborhood U_x of x and a homeomorphism $\phi : U_x \to \mathbb{R}^0$ (we called (U, ϕ) a chart). But $\mathbb{R}^0 = \{0\}$, and if $U_x \cong \{0\}$, then $U_x \cong \{x\}$. Note that this does not say that every neighborhood is one point; it says that there exists one neighborhood that is one point. So for each $x \in X$, $\{x\}$ is open, which means that X is a discrete space.[1] The connectedness of the space forces it to contain only 1 point. $\square$

So if X is a connected, closed 0-manifold, then the statement $X \simeq S^0 \implies X \cong S^0$ vacuously holds true as there does not exist such an X such that $X \simeq S^0$.

1.2 Classification of 1-manifolds

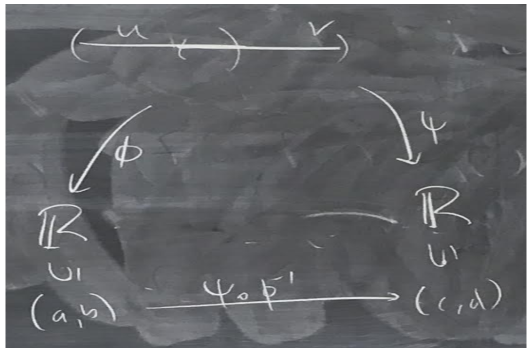
Lemma 1.1. *Let X be connected. If (U, ϕ) and (V, ψ) are charts on X and $U, V \cong \mathbb{R}$, then $U \cap V$ has at most two connected components. If $U \cap V \neq \varnothing$,*

1. *There is 1 connected component $\implies W = U \cup V \cong \mathbb{R}$.*

2. *There are 2 connected components $\implies U \cup V \cong S^1$.*

[1] In fact, every second countable (and hence countable) discrete space is a 0-manifold.

Proof. If $U \cap V \neq \varnothing$ and $U \cap V$ is connected, then $\phi(U \cap V$ and $\psi(U \cap V)$ are connected. So they are equal to (a, b) and (c, d), respectively for some $a, b, c, d \in \mathbb{R} \cup \{\pm\infty\}$ (by one of our previous theorems about connected subsets of $\mathbb{R}$). If $U \subseteq V$ or $V \subseteq U$, we are done, as $W = U$ or $W = V$. So assume neither is true, and consider $\psi \circ \phi^{-1} : (a, b) \to (c, d)$. This is a homeomorphism. Assume $\psi \circ \phi^{-1}$ is increasing (if not, replace (U, ϕ) by $(U, -\operatorname{id}_{\mathbb{R}} \circ \phi)$).



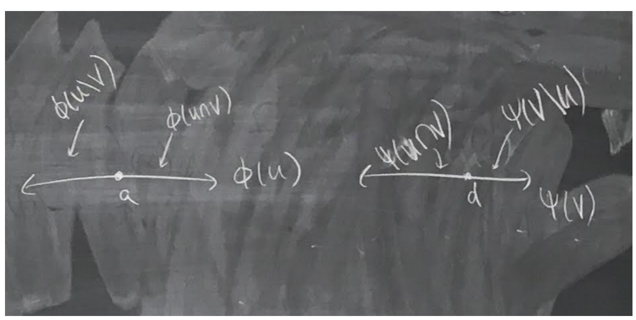
Claim: We can assume that $a \in \mathbb{R}$, $b = \infty$, $c = -\infty$, and $d \in \mathbb{R}$. If the claim is true, then assume $a < d$ (otherwise, compose ϕ with a translation). Let $f : (a, \infty) \to (a, d)$ and $g : (-\infty, d) \to (a, d)$ be homeomorphisms such that

$$(g \circ \psi)(x) = (f \circ \phi)(x) \qquad \forall x \in U \cap V.$$

Then define $\chi : U \cap V \to \mathbb{R}$ be

$$\chi(x) = \begin{cases} \phi(x) & x \in U \setminus V \\ (f \circ \phi)(x) & x \in U \cap V \\ \psi(x) & x \in V \setminus U. \end{cases}$$

Check yourself that χ is a homeomorphism.



Proof of claim: First note that $a < b \implies a \neq \infty$ and that $c < d \implies c \neq \infty$. If a, c are both finite, the consider $\tilde{a} = \phi^{-1}(a)$ and $\tilde{c} = \psi^{-1}(c)$. If $\tilde{a} \neq \tilde{c}$, then X is Hausdorff, so there exist disjoint neighborhoods $U_{\tilde{a}}, U_{\tilde{c}}$ of $\tilde{a}$ and $\tilde{c}$, respectively. So $(\psi \circ \phi^{-1})(a) = \psi(\tilde{a}) \in (c,d) \setminus \psi(U_{\tilde{c}} \cap V)$. Then $\psi \circ \phi^{-1}$ is increasing, but $(\psi \circ \phi^{-1})(a)$ is outside a neighborhood of c in (c,d). So $\psi \circ \phi^{-1}$ cannot be surjective, and $\tilde{a} = \tilde{c}$. Now $\tilde{a} = \tilde{c} \in U \cap V$, but $a = \phi(\tilde{a}) \notin \phi(U \cap V)$, which is a contradiction. So one of a, c is infinite. Similarly, only one of b, d is ∞. If $a = -\infty$ and $b = -\infty$, then $U \subseteq V$, and if $c = -\infty$ and $d = \infty$, then $V \subseteq U$. So either:

1. $a \in \mathbb{R}$, $b = \infty$, $c = \infty$, and $d \in \mathbb{R}$,

2. $a = -\infty$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, and $d = \infty$.

In the second case, just switch the names of U and V. This proves the claim.

If $U \cap V$ has 2 connected components W_1 and W_2, then since U and V are connected but $U \cap V$ is not, we must have $U \not\subseteq V$ and $V \not\subseteq U$. As above,

$$\phi(W_1) = (a,b), \qquad \phi(W_2) = (a'b'),$$

$$\psi(W_1) = (c,d), \qquad \psi(W_2) = (c',d').$$

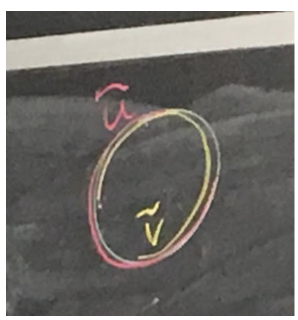
We can assume that $\phi(W_1) = (a, \infty)$ and $\phi(W_1) = (-\infty, d)$ for some $a, d, \in \mathbb{R}$. Similar analysis holds for W_2, so we conclude that

$$\phi(W_2) = (-\infty, b') \qquad \psi(W_2) = (c', \infty)$$

for some b', c' with $b' < a$ and $d < c'$. So we can write down a homeomorphism $U \cap V \to S^1$. Write $S^1 = \tilde{U} \cup \tilde{V}$, where

$$\tilde{U} = \{e^{2\pi i x} : x \in (1/4, 1)\}, \qquad \tilde{V} = \{e^{2\pi i x} : x \in (-1/4, 1/2)\}.$$

Then write a homeomorphism such that $U \to \tilde{U}$ and $V \to \tilde{V}$.



If $U \cap V$ has 3 connected components W_1, W_2, W_3, then $\phi(W_i) \subseteq \mathbb{R}$ has to be bounded for some i. But this is not possible (we skip the details due to lack of time). $\square$

We can now prove the desired classification theorem.

Theorem 1.3. *If X is a connected 1-manifold (perhaps with boundary), then X is homeomorphic to $\mathbb{R}$, S^1, $[0,1]$, or $[0,1)$.*

Proof. Pick a chart (U,ϕ) in X such that $\phi : U \to \mathbb{R}$ and such that (U,ϕ) is maximal; i.e. if (V,ψ) is another chart $\psi : V : \mathbb{R}$, then $U \cap V = \varnothing$ or has two components. If $X \ncong S^1$, then ant other V as above must be disjoint. If $X = U$, then $X \cong \mathbb{R}$. If not, there exists a point $p \in X \setminus U$ such that a chart (V,ψ) around p has $V \cap U \neq \varnothing$ (as X is connected). V must be homeomorphic to $\mathbb{R}^+$, and $U \cap V = V \setminus \{p\}$. If $X = U \cup \{p\}$, write a homeomorphism $X \cong [0,1)$.

We will redo this proof next lecture, but here is the idea. If $X \ncong [0,1)$, then $X \cong [0,1]$; otherwise, we will get a contradiction. $\square$